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An alternative approach to evaluate the average Nusselt number for mixed boundary layer conditions in parallel flow over an isothermal flat plate

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Abstract

In this paper, we present an alternative approach to evaluate the average Nusselt number for mixed boundary layer conditions in parallel flow over an isothermal flat plate. This approach can be used regardless of the critical Reynolds number where the flow transitions from laminar flow to turbulent flow. This approach is simple and uses graphical visualisation of the physical situation. This should assist comprehension and retention. It utilises the average quantity for the laminar boundary layer and the average value for turbulent boundary layer to obtain the average quantities for mixed boundary layers without the need to perform the usual integration. It can easily be incorporated into part of undergraduate chemical, mechanical and petroleum engineering curricula. A worked example is included to show the utility of the approach.

Keywords

Heat transfer coefficient, Nusselt number, isothermal flat plate.

Nomenclature

A	heat transfer area $[m^2]$
$\bar{h}_L$	average heat transfer coefficient $[W/(m^2 \cdot^o C)]$
h_x	local heat transfer coefficient at x $[W/(m^2 \cdot^o C)]$
k	fluid thermal conductivity $[W/(m \cdot^o C)]$
L	length of the plate $[m]$
$\overline{Nu}_L$	average Nusselt number correlation [–]
Pr	fluid Prandtl number [–]
Q	total heat transfer rate $[W]$
Re	Reynolds number [–]
T_∞	freestream temperature [^oC or K]
T_s	constant surface temperature [^oC or K]
u_∞	freestream velocity $[m/s]$
x	axial coordinate measure from the leading edge $[m]$
x_c	critical axial coordinate where laminar flow transitions to turbulent flow $[m]$
y	transverse coordinate $[m]$
w	width of the plate $[m]$
μ	fluid dynamic viscosity $[Pa \cdot s]$
ρ	fluid density $[kg/m^3]$

Subscripts

L	average or at $x = L$ from the leading edge
lam	laminar
$mixed$	mixed
s	surface
$turb$	turbulent
x	local at x from the leading edge
∞	freestream

Introduction

Shear stress, heat transfer and/or mass transfer for parallel flow over a flat plate is an important part of an undergraduate chemical engineering, mechanical engineering or petroleum engineering curriculum. For ease of discussion and without loss of generality, we shall present our discussion using heat transfer. The same approach can be applied to evaluate shear stress or mass transfer for the same flow arrangement.

Heat transfer from parallel flow over a flat plate forms an integral part of heat transfer coverage and is detailed in standard heat transfer [1, 2] or thermal fluids [3, 4] texts. Heat transfer rate from a flat plate is, amongst other things, a function of the average Nusselt number. The average Nusselt number with laminar boundary layer can be obtained analytically. There is no exact solution for turbulent boundary layer. The average Nusselt number for turbulent boundary layer is deduced from experimental observations of the skin friction coefficient and the modified Reynolds analogy. Undergraduate texts then proceed to evaluate the average Nusselt number for mixed boundary layer conditions by integrating these two average Nusselt numbers.

In this paper, we present an alternative approach where the Nusselt number for *mixed* boundary layer conditions is obtained using the average Nusselt number for laminar boundary layer and the average Nusselt number for turbulent boundary without the formal integration of the heat transfer coefficients for the two types of boundary layers. This approach is pictorial and can be used to evaluate the average skin friction and Sherwood number for mixed boundary layer conditions for parallel flow over a constant value (zero velocity for skin friction and constant concentration for Sherwood number) at the flat plate.

The remainder of this paper is divided into four sections. The physical situation considered in this paper is discussed in the next section. The evaluations of the Nusselt numbers for different types of boundary layers are then presented. Our approach to evaluate the Nusselt number for mixed boundary layer conditions is then explained. We then demonstrate our approach using an “advanced” problem. Some concluding remarks are given to conclude the paper.

Physical situation

In this article, we shall assume that the Newtonian fluid with constant properties. The flow is steady with negligible viscous dissipation and pressure gradient. Figure 1 shows the physical situation considered in this article. A fluid flows with a freestream velocity u_∞ and a freestream temperature T_∞ parallel to an isothermal flat plate maintained at a constant surface temperature T_s. The length of the plate is L and the width of the plate is w. There are three possible scenarios to consider when analysing the average heat transfer from a flat plate. In the first situation shown in Fig. 1a, a laminar boundary layer develops starting at the leading edge $(x = 0)$. If the plate is short such that the local Reynolds number at the trailing edge of the plate $(x = L)$ is less than the critical Reynold number $Re_{x,c}$ for the flow to transition into turbulent flow, the laminar boundary layer spans the whole length of the plate. If the length of the plate L is sufficiently long, the flow transitions to turbulent flow at x_c when the critical Reynolds number is reached as depicted in Fig. 1b. Here, we assume the flow transition abruptly from laminar into turbulent flow neglecting the transitional flow region as shown in Fig. 1b. To avoid confusion, the transition region will not be shown in subsequent figures. Figure 1c shows a situation where the flow reaches the leading edge disturbed. As a result, turbulent boundary layer starts to form and persists over the whole plate. For same flow conditions, $L_{mixed} \geq L_{lam}$. There is however, no relation between L_{turb} and the other two lengths.

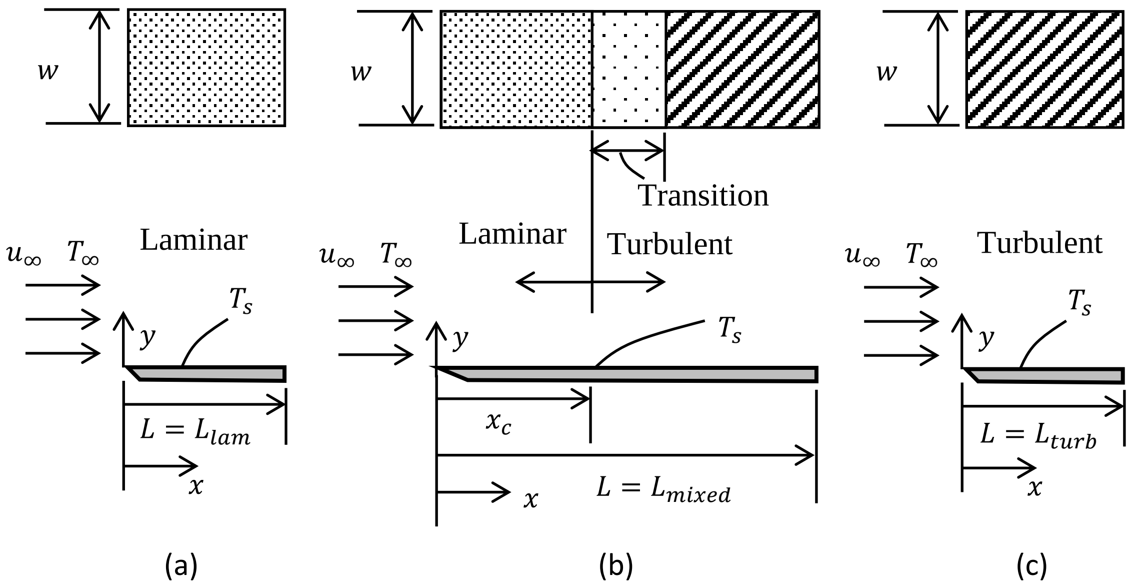


Figure 1. Top and side views of the physical situation considered (a) laminar boundary layer, (b) mixed boundary layer and (c) turbulent boundary layer.

The total heat transfer rate

Using Newton's law of cooling, the total heat transfer rate to-or-from a surface is

$$Q = \bar{h}_L A (T_s - T_\infty) \tag{1}$$

where Q is the total heat transfer rate, $\bar{h}_L$ is the average heat transfer coefficient and A is the heat transfer area. The average heat transfer coefficient for flow over a flat plate can be calculated from the Nusselt number $\overline{Nu}_L$ correlations. These two parameters are related through

$$\overline{Nu}_L \equiv \frac{\bar{h}_L L}{k} \tag{2}$$

where k is the fluid thermal conductivity. Using Eq. (2) and the area for a rectangular flat plate in Eq. (1) gives

$$Q = \overline{Nu}_L k w (T_s - T_\infty) \tag{3}$$

The total heat transfer rate from a flat plate can then be evaluated using Eq. (3) once the average Nusselt number is obtained. This is described next.

Average heat transfer coefficient and average Nusselt number

The Nusselt number correlations needed in Eq. (3) are obtained from the heat transfer coefficients. For ease of discussion and without loss of generality, let us call the *local* heat transfer coefficient at any axial location x as h_x. The average heat transfer coefficient over the whole length of the plate L can then be calculated through

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x \, dx \tag{4}$$

The average Nusselt number is then obtained using Eq. (2). The average Nusselt numbers for the laminar boundary layer and for turbulent boundary layer are described next. The common practice to evaluate the average Nusselt number for mixed boundary layer is then discussed. We shall then describe an approach to obtain the average Nusselt number for mixed boundary layer using the average Nusselt numbers for laminar and turbulent boundary layers without explicitly performing the integration over the two boundary layers.

Laminar boundary layer

When a laminar boundary layer spans the whole length of an isothermal plate, the local heat transfer coefficient for a fluid with Prandtl number $Pr \geq 0.6$ can be obtained analytically as

$$h_{x,lam} = 0.332 \frac{k}{x} Re_x^{1/2} Pr^{1/3} \tag{5}$$

where Re_x is the local Reynolds number defined as

$$Re_x \equiv \frac{\rho u_\infty x}{\mu} \tag{6}$$

In Eq. (6), ρ is the fluid density and μ is the fluid dynamic viscosity. The average Nusselt number $\overline{Nu}_L$ over the length of the plate is obtained using Eqs. (2) and (4) to (6) as

$$\overline{Nu}_{L,lam} = 0.664 Re_L^{1/2} Pr^{1/3} \tag{7}$$

where Re_L is the Reynolds number defined as[5, 6]

$$Re_L \equiv \frac{\rho u_\infty L}{\mu} \tag{8}$$

Turbulent boundary layer

As a result of the inherent unsteady nature of turbulent flow, there is no exact solution for turbulent boundary layer. From experimental observations, the local heat transfer coefficient for parallel flow over an isothermal flat plate where turbulent boundary layer prevails from the leading edge is given by

$$h_{x,turb} = 0.0296 \frac{k}{x} Re_x^{4/5} Pr^{1/3} \tag{9}$$

Using the same approach as above, the average Nusselt number can be obtained as

$$\overline{Nu}_{L,turb} = 0.037 Re_L^{4/5} Pr^{1/3} \tag{10}$$

Equation (10) is valid when $0.6 \leq Pr \leq 60$ and $Re_{x,c} \leq Re_L \leq 10^8$.[1]

Mixed boundary layer

Common approach

For the situation shown in Fig. 1b where a laminar boundary layer begins to develop at the leading edge and the length of the plate is sufficiently long for the flow to transition into turbulent flow at x_c when the critical Reynolds number $Re_{x,c}$ is reached, mixed laminar and turbulent boundary layers are encountered. For such situation, the average heat transfer coefficient is obtained from the laminar boundary layer and the turbulent boundary layer heat transfer coefficients as

$$\bar{h}_{L,mixed} = \frac{1}{L}\left(\int_0^{x_c} h_{x,lam}\,dx + \int_{x_c}^{L} h_{x,turb}\,dx\right) \tag{11}$$

Using Eqs. (5), (9) and (11), the average Nusselt number for mixed boundary layer conditions where the laminar boundary layer transitions into a turbulent boundary layer at a critical Reynolds number of $Re_{x,c}$ becomes

$$\overline{Nu}_{L,mixed} = \left(0.037Re_L^{4/5} - 0.037Re_{x,c}^{4/5} + 0.664Re_{x,c}^{1/2}\right)Pr^{1/3} \tag{12}$$

In Eq. (12), the critical Reynolds number where the flow transitions from laminar to turbulent is

$$Re_{x,c} \equiv \frac{\rho u_\infty x_c}{\mu} \tag{13}$$

When the flow transitions at $Re_{x,c} = 5 \times 10^5$, Eq. (12) reduces to the commonly known average Nusselt number correlation for the mixed boundary layer conditions of

$$\overline{Nu}_{L,mixed} = \left(0.037Re_L^{4/5} - 871\right)Pr^{1/3} \tag{14}$$

Remarks In this approach, the average heat transfer coefficient for mixed boundary layer conditions is evaluated by combining the average laminar heat transfer coefficient and the average turbulent heat transfer coefficient. Students are working with *three* sets of equations namely, one set for laminar boundary layer, a second set for turbulent boundary layer and a third set for mixed boundary layers. Once the mixed boundary layer conditions correlation (Eq. 12) is obtained, some students treat this as a formula and compute the answer without understanding the physics. Another approach is described next.

Our approach

We present an approach to evaluate the average Nusselt number for a mixed boundary conditions making use of the existing correlations for laminar boundary layer and turbulent boundary layer without carrying out the mathematical evaluation of Eq. (11).

Figure 2 shows the *top* view of a flat plate with mixed boundary conditions from which the total heat transfer rate from the entire surface must be evaluated. As indicated in Eq. (3), this can be done once we know the average Nusselt number for the whole plate. We propose to evaluate the average Nusselt number for the mixed boundary conditions using the average Nusselt number for laminar boundary layer (Eq. 7) and the average Nusselt number for turbulent boundary layer (Eq. 10) as shown in Fig. 2.

Using Fig. 2, the average Nusselt number for mixed boundary conditions is evaluated as

$$\overline{Nu}_{L,mixed} = \overline{Nu}_{L,turb} - \overline{Nu}_{x,c,turb} + \overline{Nu}_{x,c,lam} \tag{15}$$

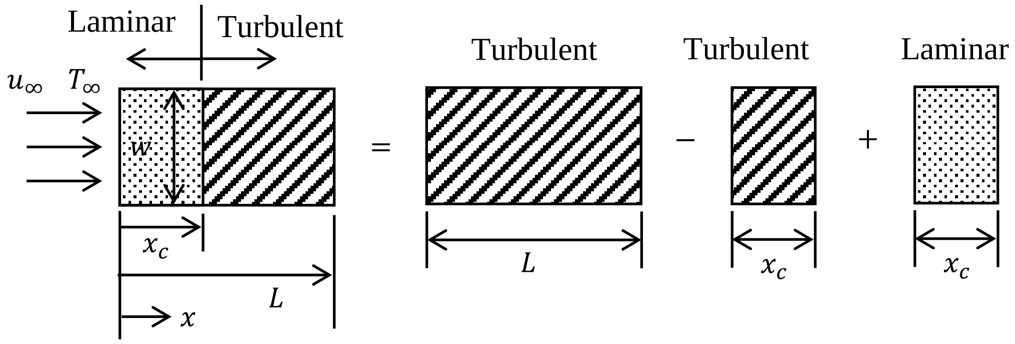


Figure 2. Top views of the physical situation considered.

Substituting Eqs. (7) and (10) with the appropriate lengths, Eq. (15) becomes

$$\overline{Nu}_{L,mixed} = 0.037Re_L^{4/5}Pr^{1/3} - 0.037Re_{x,c}^{4/5}Pr^{1/3} + 0.664Re_{x,c}^{1/2}Pr^{1/3} \qquad (16)$$

Equation (16) can be readily simplified to

$$\overline{Nu}_{L,mixed} = \left(0.037Re_L^{4/5} - 0.037Re_{x,c}^{4/5} + 0.664Re_{x,c}^{1/2}\right)Pr^{1/3} \qquad (17)$$

Comparing Eq. (17) and Eq. (12), it is clear that our approach produces the same average Nusselt number with the common approach. This is done without the formal mathematical treatment of Eq. (11).

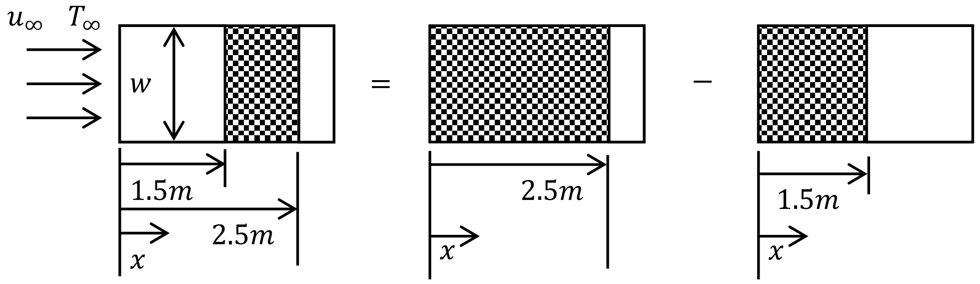
Example

In this problem, we demonstrate the use of our approach to a more "advanced" problem. We also discuss the advantage of this approach over the current practice.

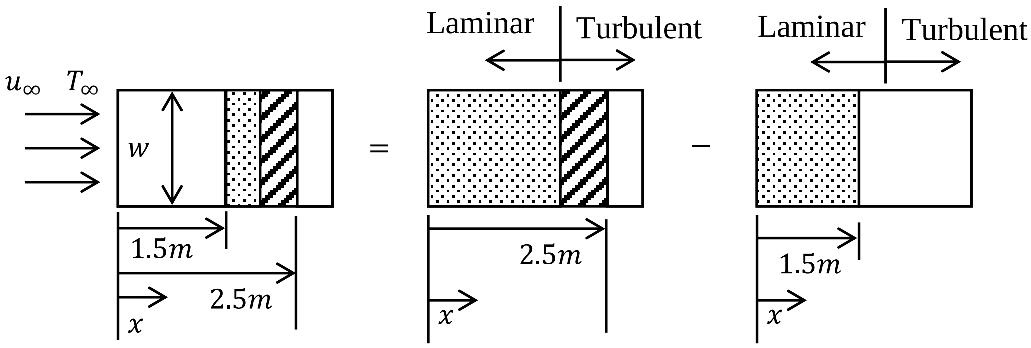
Problem statement

Air at atmospheric pressure flows parallel over a 3-m long and 1-m wide flat plate. The freestream air temperature is 300^oC while the freestream air velocity is $10\,m/s$. The plate surface temperature is kept at 30^oC. Assuming laminar boundary layer begins to develop at the leading edge of the flat plate and transitions into turbulent boundary layer at $Re_{x,c} = 6 \times 10^5$, what is the total cooling rate between $x = 1.5m$ and $x = 2.5m$ of the plate when the plate is maintained at the specified surface temperature? Air properties at the film temperature are $\rho = 0.774\,kg/m^3$, $k = 37.3 \times 10^{-3}\,W/(m \cdot K)$, $\mu = 250.7 \times 10^{-7}\,N \cdot s/m^2$, $Pr = 0.686$. Note that the standard overall Nusselt number correlation shown in Eq. (14) cannot be used as $Re_{x,c} \neq 5 \times 10^5$.

Schematic



(a)



(b)

Figure 3. Top views of the heat transfer rate between $1.5m \leq x \leq 2.5m$ (a) without boundary layer type(s) and (b) including boundary layer types.

Analysis

From Fig. 3a, we can evaluate the total heat transfer rate between $1.5m \leq x \leq 2.5m$ using

$$Q|_{x=1.5m}^{x=2.5m} = Q_{L=2.5m} - Q_{L=1.5m} \tag{18}$$

Using Newton's law of cooling, Eq. (18) can be further expanded as

$$Q|_{x=1.5m}^{x=2.5m} = (\overline{Nu}_{L=2.5m} - \overline{Nu}_{L=1.5m})kw(T_s - T_\infty) \tag{19}$$

We shall now focus on the evaluations of the average Nusselt numbers in Eq. (19). The next step is to find out what type(s) of boundary layer(s) are experienced. To do this, we shall find the location where the critical Reynolds number is reached. Using the definition of the Reynolds number (Eq. 13), we have

$$x_c = \frac{\mu}{\rho u_\infty} Re_{x,c} = \frac{250.7 \times 10^{-7}\, N \cdot s/m^2}{0.774\, kg/m^3 \times 10\, m/s} \times 6 \times 10^5 = 1.94m \tag{20}$$

As the location where the flow transition from laminar to turbulent is $1.5m \leq x_c \leq 2.5m$, mixed boundary layer conditions are encountered in the evaluations of $\overline{Nu}_{L=2.5m}$ and only laminar boundary layer prevails in the evaluation of $\overline{Nu}_{L=1.5m}$. This is depicted in Fig. 3b. The mixed boundary layers $\overline{Nu}_{L=2.5m}$ can be obtained with the approach outlined in Fig. 2 as

$$\overline{Nu}_{L=2.5m} = \left(0.037Re_{L=2.5m}^{4/5} - 0.037Re_{x,c}^{4/5} + 0.664Re_{x,c}^{1/2}\right)Pr^{1/3} \tag{21}$$

The Reynolds number $Re_{L=2.5m}$ is

$$Re_{L=2.5m} \equiv \frac{\rho u_\infty L}{\mu} = \frac{0.774\,kg/m^3 \times 10\,m/s \times 2.5m}{250.7 \times 10^{-7}\,N \cdot s/m^2} = 7.72 \times 10^5 \tag{22}$$

The Nusselt number then becomes

$$\overline{Nu}_{L=2.5m} = \left[0.037 \times (7.72 \times 10^5)^{4/5} - 0.037 \times (6 \times 10^5)^{4/5} + 0.664 \times (6 \times 10^5)^{1/2}\right] \times (0.686)^{1/3}$$

or

$$\overline{Nu}_{L=2.5m} = 760 \tag{23}$$

The laminar boundary layer $\overline{Nu}_{L=1.5m}$ is obtained from Eq. (7) as

$$\overline{Nu}_{L=1.5m} = 0.664 Re_{L=1.5m}^{1/2} Pr^{1/3} \tag{24}$$

From Eq. (24), the Reynolds number is

$$Re_{L=1.5m} \equiv \frac{\rho u_\infty L}{\mu} = \frac{0.774\,kg/m^3 \times 10\,m/s \times 1.5m}{250.7 \times 10^{-7}\,N \cdot s/m^2} = 4.63 \times 10^5 \tag{25}$$

The $\overline{Nu}_{L=1.5m}$ is then

$$\overline{Nu}_{L=1.5m} = 0.664 Re_{L=1.5m}^{1/2} Pr^{1/3} = 0.664 \times (4.63 \times 10^5)^{1/2} \times (0.686)^{1/3} = 398.5 \tag{26}$$

Using Eqs. (23) and (26) in Eq. (19), the total heat transfer rate is

$$Q|_{x=1.5m}^{x=2.5m} = (760 - 398.5) \times 37.3 \times 10^{-3}\,W/(m \cdot K) \times 1m \times (30 - 300)^oC = -3640.7W \tag{27}$$

Remarks

In this example, we showed how the proposed approach can be used in solving a more advanced problem. The pictorial approach shown in Fig. 3a is a direct extension of our approach to calculate mixed boundary layer conditions $\overline{Nu}_{L,mixed}$ in Fig. 2. Using this approach, different combinations of boundary layer types can be handled without the need to formally evaluate the Nusselt number through integrations. Initially, the students can focus on the physics of the problem and not be concerned with the formal mathematics. Once, the physics have been properly understood, the formal mathematics can then be introduced.

Concluding remarks

We present an alternative approach to evaluate the average Nusselt number for parallel flow over a flat plate maintained at constant (zero wall velocity, uniform surface temperature and uniform surface concentration) values. The approach is demonstrated using the total heat transfer rate. Applications of the approach to evaluate skin friction and mass transfer rate do not require new concepts and are straight forward. Thus, they are left to the explorations of interested readers.

Declaration of Conflicting Interests
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